Assignment Ten

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1 Section 6.1

1.1 Question Four

No. As a counterexample, suppose $n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in J$, and $m = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M(\mathbb{R})$, $mn = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \notin J$

1.2 Question Five

By Theorem 3.6, K is a subring since it is closed under subtraction: $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a-c & b-d \\ 0 & 0 \end{pmatrix}$, and absorbs products of $M(\mathbb{R})$ on the right (which is a superset of K, so K is closed under this multiplication): $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} c & d \\ e & f \end{pmatrix} = \begin{pmatrix} ac+be & ad+bf \\ 0 & 0 \end{pmatrix}$

But from the left, as a counterexample $n = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in K, m = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} \in M(\mathbb{R})$, then $mn = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \notin K$

1.3 Question Eleven

1.3.1 a

$(1) = (2) = (3) = (4) = \mathbb{Z}_5$ and $(0) = \{ 0 \}$

1.3.2 b

$(1) = (2) = (4) = (5) = (7) = (8) = \mathbb{Z}_9$, $(0) = \{ 0 \}$, and $(3) = (6) = \{ 0, 3, 6 \}$

1.3.3 c

$(1) = (5) = (7) = (11) = \mathbb{Z}_{12}$, $(2) = (6) = (10) = \{ 0, 2, 4, 6, 8, 10 \}$, $(4) = (8) = \{ 0, 4, 8 \}$, $(3) = (9) = \{ 0, 3, 6, 9 \}$, and $(6) = \{ 0, 6 \}$

1.4 Question Thirteen

No, $\mathbb{Z}_5$ is a commutative ring, but in question above, $(1) = (2)$ but $1 \neq 2$.

1.5 Question Sixteen

1.5.1 a

If we can show that $(4, 6)(2)$ and $(2)(4, 6)$, then we can conclude that $(4, 6) = (2)$
Each element $x \in (4, 6) \Rightarrow x = 4m + 6n$ for $m, n \in \mathbb{Z}$.
Thus $x = 2(2m + 3n)$ which shows that x is a multiple of two, and thus, $(4, 6)(2)$.
Each element $y \in (2) \Rightarrow y = 2p$ for $p \in \mathbb{Z}$. When p is itself a multiple of two, this can be rewritten as $y = 2(2o) = 4o$ and thus $y \in (4, 6)$.
When y is odd, $y = 2(2(q - 1) + 3) = 4q + 6$ for some $q \in \mathbb{Z}$. Therefore, y is in $(4, 6)$.

1.5.2 b

We'll take the same approach as a:
Each element $x \in (6, 9, 15) = 6m + 9n + 15o$ with $m, n, o \in \mathbb{Z}$.
Thus $x = 3(2m + 3n + 5o)$ is a multiple of three, so $x \in (3)$.
Each element $y \in (3) \Rightarrow y = 3p$ for $p \in \mathbb{Z}$. When p is a multiple of two, then $y = 3(2q) = 6q \in (6, 9, 15)$.
When p is odd, $y = 3(2(u - 1) + 7)$ for some $u \in \mathbb{Z}$, $y = 6u + 15 \in (6, 9, 15)$.

1.6 Question Seventeen

1.6.1 a

If $a \in I \cap J$, and $b \in I \cap J$, then $a \in I$, $b \in I$, $a \in J$, and $b \in J$. Then, $a - b \in I$ and $a - b \in J$, so $a - b \in I \cap J$.
If $a \in I \cap J$ and $r \in R$, then $ra \in I$, $ra \in J$, $ar \in J$, and $ar \in I$ by definition of I and J.
Therefore, $I \cap J$ is an ideal in R.