Assignment Eleven

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April 17, 2023

1 Section 6.1

1.1 Question Twenty-One

The ideal $(a)+(b)$ is generated by all linear combinations of a and b, $x \in (a)+(b) \Rightarrow x = ac_1+bc_2$
By definition, $d|a$ and $d|b$, so $a = da_1$ and $b = db_1$. Then, $x \in (a) + (b) \Rightarrow x = da_1c_1 + db_1c_2 = d(a_1c_1 + b_1c_2)$, so $(a) + (b)(c)$.
By Theorem 1.2, $d = au + bv$ which is a linear combination of a and b, so $d \in (a) + (b)$. Since every multiple of d is also a multiple of a and b, $(d)(a) + (b)$.
So, $(d) = (a) + (b)$.

2 Section 6.2

2.1 Question Two

Let $f : F \to R$ with R being the image, so the kernel of f is an ideal in the field F by Theorem 6.10. Then assuming from the hint, that question ten in 6.1 is true, the only ideals in F are either (0_F) or F itself. So, $\ker(f) = (0_F)$ or F.
When $\ker(f) = (0_F)$ then f is injective by Theorem 6.11, and is thus an isomorphism.
When $\ker(f) = F$ then $R = \{0_F\}$.

2.2 Question Four

2.2.1 a

f is consistent, as when $[a]_{12} = [b]_{12}(a \equiv b \bmod 12 \Rightarrow a - b = 12n)$, then $[a]_4 = [b]_4$ as $a - b \equiv 20n \bmod 4 = a - b \equiv 0 \bmod 4 \Rightarrow a \equiv b \bmod 4$.

2.2.2 b

The kernel of f is the ideal (4) as $f(x) \ni x \in (4) \Rightarrow f(x) = [4n]_4 = [0]_4$.

2.3 Question Six

The kernel of ϕ are polynomials in $\mathbb{R}[x]$ with a root of 2. By Theorem 4.16, $x - 2$ must be a factor, so $\ker(\phi)$ is the set of all multiples of $x - 2$ in $\mathbb{R}[x] = (x - 2)$.

2.4 Question Nine

2.4.1 a

There is surjectivity as $x \in \mathbb{Z}$, then the matrix $\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right)$ maps to x.
It is a homomorphism since $f(x + y) = \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) + \left(\begin{smallmatrix} e & 0 \\ g & h \end{smallmatrix}\right) = \left(\begin{smallmatrix} a+e & 0 \\ c+g & d+h \end{smallmatrix}\right) = (a + e) = f(x) + f(y)$, $f(xy) = \left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} e & 0 \\ g & h3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} ae & 0 \\ ce+dc & dh \end{smallmatrix}\right) = ae = f(x)f(y)$, and the identity matrix is mapped to 1.

2.4.2 b

$\{\left(\begin{smallmatrix} 0 & 0 \\ c & d \end{smallmatrix}\right) | c, d \in \mathbb{Z}\}$

2.5 Question Ten

2.5.1 a

By definition, $x, y \in f(I) \Rightarrow \exists z, t \in I \ni f(z) = x, f(t) = b$, and $f(I)$ is also a subset of S.
By hormomorphism, $f(z) - f(t) = x - y \Rightarrow f(z - t) = x - y \in I$.
For $s \in S \Rightarrow \exists r \in R \ni f(r) = s$ by surjection. Then, $xf(r) = sx = f(z)f(r) = f(zr) \in f(I)$

2.5.2 b

Let $R = \mathbb{R}$, $S = \mathbb{C}$, and $f : R \to S \ni f(x) = x + 0i$. f is not a surjective homomorphism (i has no associated element in $\mathbb{R}$), and $\mathbb{R}$ is an ideal in $\mathbb{R}$.
However, $f(\mathbb{R})$ is not an ideal by the trivial example that $f(1)(i) = (1 + 0i)(0 + i) = i \notin \mathbb{R}$.

2.6 Question Seventeen

2.6.1 a

$f(a+b) = ((a+b)+I, (a+b)+J) = ((a+I)+(b+I), (a+J)+(b+J)) = (a+I, a+J)+(b+I, b+J) = f(a) + f(b)$.
Similarly, $f(a)f(b) = f(ab)$: $f(ab) = (ab+I, ab+J)$ and $f(a)f(b) = (a+I, a+J)(b+I, b+J) = ((a + I)(b + I), (a + J)(b + J)) = (ab + I, ab + J)$.

2.6.2 b

If we consider $R = \mathbb{Z}, I = (2)$, and $J = (4)$, the elements in Z/(2) x Z/(4) are:
{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (1, 3)}
There's at least one domain element that is impossible to get to: $(1, 0)$ - an integer can't be equivalent to 1 mod 2 and also 0 mod 4. So, f is not necessarily surjective.

2.7 Question Twenty-One

Let $f : \mathbb{Z}_{20} \to \mathbb{Z}_5$ such that $f([a]_{20}) = [a]_5$. f is consistent because when $[a]_{20} = [b]_{20}(a \equiv b \bmod 20 \Rightarrow a - b = 20n)$, then $[a]_5 = [b]_5$ as $a - b \equiv 20n \bmod 5 = a - b \equiv 0 \bmod 5 \Rightarrow a \equiv b \bmod 5$.
Also, f is a surjective homomorphism:

- $f([a]_{20} + [b]_{20}) = f([a + b]_{20}) = [(a + b)]_5 = [a]_5 + [b]_5 = f([a]_{20}) + f([b]_{20})$
- $f([a]_{20} * [b]_{20}) = f([a * b]_{20}) = [(a * b)]_5 = [a]_5 * [b]_5 = f([a]_{20}) * f([b]_{20})$
- For each $[x]_5$ there exists $[y]_{20}$ such that $f([y]_{20}) = [x]_5$. The trivial solution is that $x = y$.

Finally, the kernel of f is the ideal (5) as $x \in (5) \Rightarrow f(x) = [5n]_5 = [0]_5$. Thus, $\mathbb{Z}_{20}/(5)$ is isomorphic to $\mathbb{Z}_5$ by the First Isomorphism Theorem.