Midterm 2

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1 Question One

In $\mathbb{Z}_3[x]$, $4x^3 + 4x + 4 \equiv x^3 + x + 1$

$$
\begin{array}{rl}
 & \frac{1}{2}x \\ \cline{2-2}
2x^2+1) & x^3 \quad + x + 1 \\
 & - x^3 - \frac{1}{2}x \\ \cline{2-2}
 & \frac{1}{2}x + 1
\end{array}
$$

$\frac{1}{2}x \equiv_3 2x$ and $\frac{1}{2}x + 1 \equiv_3 (2x + 1)$
So, $a(x) = b(x)q(x) + r(x) = (2x^2 + 1)(2x) + (2x + 1)$

2 Question Two

$$
\begin{array}{rl}
 & x^3 + 3x^2 \quad + 6x + 13 \\ \cline{2-2}
x-2) & x^4 \quad + x^3 \qquad\qquad + x \quad + 1 \\
 & - x^4 + 2x^3 \\ \cline{2-2}
 & 3x^3 \\
 & - 3x^3 + 6x^2 \\ \cline{2-2}
 & 6x^2 \quad + x \\
 & - 6x^2 + 12x \\ \cline{2-2}
 & 13x \quad + 1 \\
 & - 13x + 26 \\ \cline{2-2}
 & 27
\end{array}
$$

Shows that the remainder will be zero when we are in $\mathbb{Z}_3[x]$

3 Question Three

We will use the Euclidean algorithm to find the GCD:

3.1 GCD

$$
\begin{aligned}
3x^3 + 2x^2 + 2x + 3 &= (3x - 1)(x^2 + x + 3) + (4x + 1) \\
(x^2 + x + 3) &\equiv_5 (4x + 1)(4x + 3) + 45 \equiv_5 (4x + 1)(4x + 3) + 0
\end{aligned}
$$

So, the GCD is (4x + 1)

3.2 (supplement) Division Algorithm Work

$$\begin{array}{r} 3x-1 \\ x^2+x+3\,\overline{\big)\; 3x^3+2x^2+2x+3} \\ -3x^3-3x^2-9x \phantom{{}+3} \\ \hline -x^2-7x+3 \\ x^2\;\;+x+3 \\ \hline -6x+6 \end{array}$$

$$\begin{array}{r} \frac{1}{4}x+\frac{3}{16} \\ 4x+1\,\overline{\big)\; x^2\;\;+x\;\;+3} \\ -x^2-\frac{1}{4}x \phantom{{}+3} \\ \hline \frac{3}{4}x\;\;+3 \\ -\frac{3}{4}x-\frac{3}{16} \\ \hline \frac{45}{16} \end{array}$$

Check the GCD:

$$\begin{array}{r} \frac{3}{4}x^2+\frac{5}{16}x+\frac{27}{64} \\ 4x+1\,\overline{\big)\; 3x^3+2x^2\;\;+2x\;\;+3} \\ -3x^3-\frac{3}{4}x^2 \phantom{{}+2x+3} \\ \hline \frac{5}{4}x^2\;\;+2x \phantom{{}+3} \\ -\frac{5}{4}x^2-\frac{5}{16}x \phantom{{}+3} \\ \hline \frac{27}{16}x\;\;+3 \\ -\frac{27}{16}x-\frac{27}{64} \\ \hline \frac{165}{64} \end{array}$$

$\frac{3}{4}x^2+\frac{5}{16}x+\frac{27}{64} \equiv_5 2x^2+3$ and $\frac{165}{64} \equiv_5 0$

$$\begin{array}{r} \frac{1}{4}x+\frac{3}{16} \\ 4x+1\,\overline{\big)\; x^2\;\;+x\;\;+3} \\ -x^2-\frac{1}{4}x \phantom{{}+3} \\ \hline \frac{3}{4}x\;\;+3 \\ -\frac{3}{4}x-\frac{3}{16} \\ \hline \frac{45}{16} \end{array}$$

$\frac{1}{4}x+\frac{3}{16} \equiv_5 4x+3$ and $\frac{45}{16} \equiv_5 0$.

4 Question Four

4.1 a

$\mathbb{Z}_3[x]$ is a field by Theorem 2.8.
There are no roots of $2x^3+x+1$ in $\mathbb{Z}_3$, so by Theorem 5.10 $\mathbb{Z}_3[x]/(2x^3+x+1)$ is a field, as it is irreducible by Corollary 4.19.

4.2 b

Since we've proven $\mathbb{Z}_3[x]/(2x^3+x+1)$ is a field, we know that $[2x+1]$ must be a unit.
By similar reasoning to the proof of Theorem 5.10 (1), $gcd(2x+1, 2x^3+x+1) = 1$
Then,

$$
\begin{array}{r l}
 & \qquad\qquad x^2 - \frac{1}{2}x + \frac{3}{4} \\ \cline{2-2}
2x+1 \big) & 2x^3 \qquad\quad + x + 1 \\
 & -2x^3 - x^2 \\ \cline{2-2}
 & \qquad -x^2 + x \\
 & \qquad x^2 + \frac{1}{2}x \\ \cline{2-2}
 & \qquad\qquad \frac{3}{2}x + 1 \\
 & \qquad\qquad -\frac{3}{2}x - \frac{3}{4} \\ \cline{2-2}
 & \qquad\qquad\qquad \frac{1}{4}
\end{array}
$$

Shows that:

$$
\begin{aligned}
2x^3 + x + 1 &= (2x+1)(x^2 + x) + 1 \\
1 &= (2x^3 + x + 1) + (2x+1)(-x^2 - x) \\
1 - (2x^3 + x + 1) &= (2x+1)(-x^2 - x) \\
1 &\equiv_{2x^3+x+1} (2x+1)(-x^2 - x)
\end{aligned}
$$

and thus, by similar logic again found in the proof mentioned before, $[-x^2 - x] = [2x^2 + 2x]$ must be the inverse.
As a sanity check,

$$
\begin{array}{r l}
 & \qquad\qquad\qquad x - 1 \\ \cline{2-2}
2x^2+2x \big) & 2x^3 \qquad\quad + x + 1 \\
 & -2x^3 - 2x^2 \\ \cline{2-2}
 & \qquad -2x^2 + x \\
 & \qquad 2x^2 + 2x \\ \cline{2-2}
 & \qquad\qquad\quad 3x + 1
\end{array}
$$

shows that the remainder is a unit in $\mathbb{Z}[3]$.

5 Question Five

Firstly, $\mathbb{Z}_5[x]$ is a field by Theorem 2.8.

5.1 a

There are no roots of $x^2 + 2x + 3$ in $\mathbb{Z}_5$, so by Theorem 5.10 it is irreducible.
$f_1 : (0, 1, 2, 3, 4) \to (3, 1, 1, 3, 2)$

5.2 b

This is reducible since 1 is a root.

$$
\begin{array}{r l}
 & \qquad x^2 + x + 2 \\ \cline{2-2}
x-1 \big) & x^3 \qquad + x + 3 \\
 & -x^3 + x^2 \\ \cline{2-2}
 & \qquad x^2 + x \\
 & \quad -x^2 + x \\ \cline{2-2}
 & \qquad\quad 2x + 3 \\
 & \qquad -2x + 2 \\ \cline{2-2}
 & \qquad\qquad\quad 5
\end{array}
$$

Thus, $f_2(x) = (x+4)(x^2+x+2)$

6 Question Six

Following pages 137 - 138 of the book...
By Corollary 4.19 we know (x^2+1) is irreducible in $\mathbb{R}[x]$ since its only roots are in $\mathbb{C}$: $\pm i$.
Because of Corollary 5.12, we know that there is an extension field K that contains a root to (x^2+1). Namely, some $\alpha = [x]$. By Corollary 5.5, each element can be rewritten as $[ax+b]$ with $a, b \in \mathbb{R}$. We can create a mapping f such that $[a+bx] \to [a]+[b][x] = a+b\alpha$ is unique and $f^{-1}(a+b\alpha) \to [a+bx]$ (bijection).
Additionally, we can show that $f([a+bx])+f([c+dx]) = (a+b\alpha)+(c+d\alpha) = (a+c)+(b+d)\alpha = f([(a+c)x+(b+d)])$.

$$[ax+b][cx+b] = [acx^2 + abx + bcx + bd]$$

$$\begin{array}{r|l} & \hfill ac \\ \hline x^2+1 & acx^2 + (1ab+1bc)\,x \hfill + bd \\ & -\,acx^2 \hfill -\,ac \\ \cline{2-2} & \hfill (1ab+1bc)\,x + (-1ac+1bd) \end{array}$$

So, multiplication in K is given:

$$[ax+b][cx+b] = [(ab+bc)x + ac - bd]$$

Then, we can show that by definition of α, $f([a+bx]) \cdot f([c+dx]) = (a+b\alpha)\cdot(c+d\alpha) = (ac+ad\alpha+bc\alpha+bd\alpha^2) = (ac-bd+(ad+bc)\alpha) = f([(ab+bc)x+ac-bd]) = f([a+bx][c+dx])$.
Now, replacing our work with $\alpha = i$, f is an isomorphism from $\mathbb{R}/(x^2+1)$ to $\mathbb{C}$ since it is a bijection and satisfies the addition and multiplication rules.