

Assignment Four

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1 Section 3.1

1.1 Question One

1.1.1 a

$$a \in R, b \in R \Rightarrow a + b \in R$$

1.1.2 b

$$a \in R, x \in R \ni a + x = 0_R$$

1.2 Question Three

1. All operations are closed since only the elements in $0, e, a, b$ appear in the tables.
2. From the second row and columns in the multiplication table we see that e is the multiplicative identity.
3. From the first row and columns in the addition table we see that 0 is the zero element.
4. In this field, each element is its own additive inverse.
5. There is commutativity as the transpose of each table is identical to the original (symmetry along the diagonals).

1.3 Question Six

1.3.1 a

Since our addition and multiplication operators are the same in $\mathbb{Z}$, we have associativity, commutativity, and distributivity.

Sums of multiples of 3 are also multiples of 3: $3n + 3m = 3(n + m)$, so this set is closed under addition.

Products of multiples of 3 are also multiples of 3: $(3n)(3m) = (3)(3nm)$, so this set is closed under multiplication.

The additive inverse exists in the set for every element: $3n + x = 0 \Rightarrow x = 3 \cdot (-n)$.

0 is the zero element and is a multiple of 3 since $3 \cdot 0 = 0$.

Therefore $x : x = 3n \ni n \in \mathbb{Z}$ is a subring of $\mathbb{Z}$

1.3.2 b

Since our addition and multiplication operators are the same in $\mathbb{Z}$, we have associativity, commutativity, and distributivity.

Sums of multiples of k are also multiples of k : $kn + km = k(n + m)$, so this set is closed under addition.

Products of multiples of k are also multiples of k : $(kn)(km) = (k)(knm)$, so this set is closed under multiplication.

The additive inverse exists in the set for every element: $kn + x = 0 \Rightarrow x = k \cdot (-n)$.

0 is the zero element and is a multiple of k since $k \cdot 0 = 0$.

Therefore $x : x = kn \ni n \in \mathbb{Z}$ is a subring of $\mathbb{Z}$

1.4 Question Nine

1.4.1 a

$(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)$

1.4.2 b

Since our addition and multiplication operators are the same as in R , then we have associativity, commutativity, and distributivity.

Sums of elements in R^* are also in R^* since any element $(r, r) \in R^*, (j, j) \in R^* \Rightarrow (r + j, r + j) \in R^*$.

Products of elements in R^* are also in R^* since any element $(r, r) \in R^*, (j, j) \in R^* \Rightarrow (rj, rj) \in R^*$.

The additive inverse exists in the set for every element: $(r, r) + x = 0 \Rightarrow x = (-r, -r) \in R^*$.

$(0, 0) \in R^*$ is the zero element.

Therefore $(r, r) : (r, r) \in R^*$ is a subring of R^* .

1.5 Question Ten

Consider $a \in S \ni a = (10, -10)$ and $b \in S \ni b = (10, -10)$, then $ab \notin S$ since $ab = (100, 100)$ which does not follow the rule that $100 + 100 = 0$.

1.6 Question Eleven

1.6.1 a

Addition is closed since $\begin{pmatrix} a & a \\ b & b \end{pmatrix} + \begin{pmatrix} c & c \\ d & d \end{pmatrix} = \begin{pmatrix} a+c & a+c \\ b+d & b+d \end{pmatrix}$ which is of the form of the given rule.

It is also closed under multiplication since $\begin{pmatrix} a & a \\ b & b \end{pmatrix} \cdot \begin{pmatrix} c & c \\ d & d \end{pmatrix} = \begin{pmatrix} ac+ad & ac+ad \\ bc+bd & bc+bd \end{pmatrix}$ which is also of the form of the given rule.

The zero element is the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, trivially.

Associativity and commutivity (of addition) come from these operations existing for 2x2 matrices in $M(\mathbb{R})$.

1.6.2 b

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a(1)+a(0) & a(1)+a(0) \\ b(1)+b(0) & b(1)+b(0) \end{pmatrix} \text{ which is equivalent to } \begin{pmatrix} a & a \\ b & b \end{pmatrix}$$

1.6.3 c

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} ((1)(1)+(1)(2)) & ((1)(1)+(1)(2)) \\ ((0)(1)+(0)(2)) & ((0)(1)+(0)(2)) \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}.$$

1.7 Question Twelve

$\mathbb{Z}[i]$ is closed under addition: $(a+bi) + (c+di) = (a+c) + (b+d)i$ and since $\mathbb{Z}$ is a ring itself, $(a+c) + (b+d)i$ is also in $\mathbb{Z}[i]$.

$\mathbb{Z}[i]$ is closed under multiplication: $(a+bi) \cdot (c+di) = ac + adi + cbi + bdi^2 = (ac - bd) + (ad + bc)i$ by similar logic.

The additive inverse always exists in the set: $(a+bi) + x = 0 \Rightarrow x = -a - bi$. Finally, the zero element is trivially $0 + 0i$ since $(a+bi) + (0+0i) = a+bi$.

1.8 Question Fourteen

The zero element is given as 2.

S is closed under addition. For example given some $f, g \in S$ then $(f+g)(x) = h$ and h will still satisfy the rule that $h(2) = 0$ since $(f+g)(2) = f(2) + g(2) = 0 + 0$, and addition was already closed in the domain given that question 8 is a ring itself.

Similarly, S is closed under multiplication. $(f \cdot g)(x) = h$ and h will still satisfy the rule as $h(2) = f(2) \cdot g(2) = 0 \cdot 0$.

The zero element is $f \ni f(x) = 0$.

Finally, the additive inverse g exists in the set for each f since $f + g = 0$ implies that g is just $-f$ (the rule still stands here).

1.9 Question Fifteen

1.9.1 b

$\odot$	(0, 0)	(0, 1)	(0, 2)	(1, 0)	(1, 1)	(1, 2)
(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)	(0, 0)
(0, 1)	(0, 0)	(0, 1)	(0, 2)	(0, 0)	(0, 1)	(0, 2)
(0, 2)	(0, 0)	(0, 2)	(0, 1)	(0, 0)	(0, 2)	(0, 1)
(1, 0)	(0, 0)	(0, 0)	(0, 0)	(1, 0)	(1, 0)	(1, 0)
(1, 1)	(0, 0)	(0, 1)	(0, 2)	(1, 0)	(1, 1)	(1, 2)
(1, 2)	(0, 0)	(0, 2)	(0, 1)	(1, 0)	(1, 2)	(1, 1)

$\oplus$	(0, 0)	(0, 1)	(0, 2)	(1, 0)	(1, 1)	(1, 2)
(0, 0)	(0, 0)	(0, 1)	(0, 2)	(1, 0)	(1, 1)	(1, 2)
(0, 1)	(0, 1)	(0, 2)	(0, 0)	(1, 1)	(1, 2)	(1, 0)
(0, 2)	(0, 2)	(0, 0)	(0, 1)	(1, 2)	(1, 0)	(1, 1)
(1, 0)	(1, 0)	(1, 1)	(1, 2)	(2, 0)	(2, 1)	(2, 2)
(1, 1)	(1, 1)	(1, 2)	(1, 0)	(2, 1)	(2, 2)	(2, 0)
(1, 2)	(1, 2)	(1, 0)	(1, 1)	(2, 2)	(2, 0)	(2, 1)

1.10 Question Eighteen

No, by definition, as the distributive axiom is violated for this to be a ring. For example, $1(1 + 1) = (1)(1) + (1)(1) = 2$ but $1(1 + 1) = 1$.