Assignment Seven

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1 Section 4.2

1.1 Question Five

1.1.1 b

$$\begin{aligned} x^5+x^4+2x^3-x^2-x-2 &= (x-1)(x^4+2x^3+5x^2+4x+4)+(-x^3-x+2) \\ x^4+2x^3+5x^2+4x+4 &= (-x-2)(-x^3-x+2)+(4x^2+4x+8) \\ -x^3-x+2 &= (\frac{-x}{4})(4x^2+4x+8)+0 \end{aligned}$$

$(4x^2+4x+8) = 4(x^2+x+2)$
x^2+x+2

1.1.2 c

$deg(d) = 2$ is the greatest degree of a possible common divisor, so we'll stick with x^2-1.

1.1.3 g

$$\begin{aligned} 2x^4+5x^3-5x-2 &= (x+4)(2x^3-3x^2-2x)+(14x^2+3x-2) \\ 2x^3-3x^2-2x &= (\frac{x}{7}-\frac{12}{49})(14x^2+3x-2)+(\frac{-48x-24}{49}) \\ 14x^2+3x-2 &= (\frac{-7x}{24})(\frac{-48x-24}{49})+0 \end{aligned}$$

$\frac{-48x-24}{49} = \left(\frac{49}{-48}\right)\left(\frac{(-48x-24)}{49}\right) = x+\frac{1}{2}$
$x+\frac{1}{2}$

1.2 Question Ten

$x^3-3abx+a^3+b^3$ can be factored to $(a+b+x)(a^2-ab-ax+b^2-bx+x^2)$, so $a+b+x$ is the gcd.

2 Section 4.3

2.1 Question Three

2.1.1 a

$\{\ x^2+x+1,\ 2x^2+2x+2,\ 3x^2+3x+3,\ 4x^2+4x+4\ \}$

2.1.2 b

$\{\ 3x+2,\ 6x+4,\ 2x+6,\ 5x+1,\ x+3,\ 4x+5\ \}$

2.2 Question Six

Assume that x^2+1 is in fact reducible. Then, $x^2+1=(ax+b)(cx+d)$. Thus, $ax\cdot cx=x^2\Rightarrow ac=1$, $axd+bcx=0\Rightarrow ad+bc=0$, and $bd=1$ with a,b,c,d all being nonzero.
Then, $a=\frac{1}{c}$ and $d=\frac{1}{b}$, so $ad=-bc\Rightarrow\frac{1}{cb}=-bc$, which is impossible.

2.3 Question Nine

2.3.1 a

x^2+x+1

2.3.2 b

x^3+x^2+1 and x^3+x+1

2.4 Question Ten

2.4.1 a

In $\mathbb{Q}[x]$, no. In $\mathbb{R}[x]$, yes: $x^2-3=(x-\sqrt{3})(x+\sqrt{3})$.

2.4.2 b

Yes, in both fields: $x^2+x-2=(x+2)(x-1)$

2.5 Question Eleven

Assume that x^3-3 were reducible in $\mathbb{Z}_7[x]$. Then, there must be a monic factor of degree one, as $x^3-3=g(x)h(x)\Rightarrow deg(x^3-3)=deg(g(x)h(x))\Rightarrow 3=deg(g(x))+deg(h(x))$ by Theorem 4.2, and either $deg(g(x))$ or $deg(h(x))$ must be one, with the other, two.
So, one of the factors must be of the form $x+a, a\in\mathbb{Z}_7$. None such factor exists since polynomial division always returns a non-zero remainder:
$x\nmid x^3-3$, trivially.
$\frac{x^3-3}{x+1}$ gives a remainder of 4.
$\frac{x^3-3}{x+2}$ gives a remainder of $-11\equiv_7 3$.
$\frac{x^3-3}{x+3}$ gives a remainder of $-30\equiv_7 5$.
$\frac{x^3-3}{x+4}$ gives a remainder of $-67\equiv_7 3$.
$\frac{x^3-3}{x+5}$ gives a remainder of $-128\equiv_7 5$.
$\frac{x^3-3}{x+6}$ gives a remainder of $-219\equiv_7 5$.

2.6 Question Twelve

In $\mathbb{Q}[x]$, $x^4-4=(x^2-2)(x^2+2)$
In $\mathbb{R}[x]$, $x^4-4=(x-\sqrt{2})(x+\sqrt{2})(x^2+2)$
In $\mathbb{C}[x]$, $x^4-4=(x-\sqrt{2})(x+\sqrt{2})(x-i\sqrt{2})(x+i\sqrt{2})$

2.7 Question Fourteen

$x^2+x\equiv_6(x+4)(x+3)$
$x^2+x\equiv_6 x(x+1)$

3 Section 4.4

3.1 Question Two

3.1.1 c

By Theorem 4.15, the remainder is $f(-1) = 5$

3.2 Question Three

3.2.1 c

If the remainder of $\frac{f(x)}{h(x)}$ is 0, then h is a factor of f, so using Theorem 4.15 we can find that $f(-2) = -55 \equiv_5 0$, so h is indeed a factor.

3.3 Question Four

3.3.1 b

We need to find k such that $f(-1) = 0(mod5)$

```
>>> f = lambda k,x: (x**4 + 2 * x**3 - 3 * x**2 + k * x + 1) % 5
>>> for i in range(5):
...   if f(i, -1) == 0:
...     print(i)
...
2
```

$k = 2$ works nicely!

3.4 Question Seven

```
>>> set(filter(lambda x: (x**7 - x) % 7 == 0, range(7)))
{0, 1, 2, 3, 4, 5, 6}
```

Shows that each element is a root, so the factoring is correct by the Factor Theorem.

3.5 Question Eight

3.5.1 b

The polynomial is irreducible since its only roots are $\pm\sqrt{7} \notin \mathbb{Q}$, by Corollary 4.19

3.5.2 d

```
>>> set(filter(lambda x: (2 * x**3 + x**2 + 2 * x + 2) % 5 == 0, range(5)))
set()
```

It's irreducible since there are no roots in $\mathbb{Z}_5$.

3.6 Question Nine

```
>>> def find_irr_mon_polys(deg, mod):
...     z_s = range(mod)
...     polys = set()
...     for b in z_s:
...        for c z_s:
...           f_repr = f"x^2 + {b}x + {c}"
...           f = lambda x: (x**2 + b*x + c) % mod
...           if not any(map(lambda x: f(x) == 0, z_s)):
...               polys.add(f_repr)
...     return polys
```

```
>>> find_irr_mon_polys(2, 5)
```

$\{\ x^2 + 0x + 2, x^2 + 0x + 3, x^2 + 1x + 1, x^2 + 1x + 2, x^2 + 2x + 3, x^2 + 2x + 4, x^2 + 3x + 3, x^2 + 3x + 4, x^2 + 4x + 1, x^2 + 4x + 2\ \}$

```
>>> find_irr_mon_polys(2, 3)
```

$\{\ x^2 + 0x + 1, x^2 + 1x + 2, x^2 + 2x + 2\ \}$

3.7 Question Thirteen

3.7.1 a

If $f(x) = cg(x)$ with $c \neq 0_F$, then $g(x) = c^{-1}f(x)$ and $f(x) = c^{-1}g(x)$. Then, $g(y) = 0_F \Leftrightarrow f(y) = 0_F$.

3.7.2 b

No, consider $f(x) = x$ and $g(x) = x^2$ in $\mathbb{Z}$, then f and g share 0 as their only root, but they are not associates.