Midterm One

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1 Question One

```
In [1]: sols = lambda n: set(filter(lambda x: (x**2 + x) % n == 0, range(n)))

In [2]: sols(5)
Out[2]: {0, 4}

In [3]: sols(6)
Out[3]: {0, 2, 3, 5}
```

2 Question Two

$p|a \Rightarrow np = a$
$p|a + bc \Rightarrow mp = a + bc \Rightarrow mp = np + bc \Rightarrow (m - n)p = bc$ which implies p is a factor of bc ($p|bc$), so by Theorem 1.5, $p|b$ or $p|c$.

3 Question Three

From the multiplication table we can see that e is the identity element 1_R, the zero element 0_R is z. From the addition table we find that every element is its own additive inverse.

3.1 a

The units in this ring are e, g, and the zero divisors are a, b, c, f.

3.2 b

$3bd = 3(bd) = 3z = b$
$2ac = 2(ac) = 2b = b + b = z$
$g^{-1}f^3 = (g)(fff)(gf)(ff) = (d)(f) = d$
Thus, $3bd - 2ac + g^{-1}f^3 = b - z + d = b + d = f$.

4 Question Four

To get a hunch,

```
In [1]: import numpy as np

In [2]: def find_counter(matrix_generator, in_ring, search_space=range(-100, 100)):
...:     for a1 in search_space:
...:         n = matrix_generator(a1)
```

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...:         for a2 in search_space:
...:             m = matrix_generator(a2)
...:             dot = np.dot(n, m)
...:             add = np.add(n, m)
...:
...:             if not in_ring(dot):
...:                 return (n, m, dot, '*')
...:             if not in_ring(add):
...:                 return (n, m, add, '+')
...:
...:     return None
...:

In [3]: find_counter(lambda a: [[0, a], [0, -a]], \
                     lambda m: m[0][1] == -m[1][1] and m[0][0] == 0 \
                     and m[1][0] == 0)

In [4]: find_counter(lambda a: [[0, a], [a, 0]], \
                     lambda m: m[0][1] == m[1][0] and m[1][1] == 0 \
                     and m[0][0] == 0)
Out[4]:
([[0, -100], [-100, 0]],
 [[0, -100], [-100, 0]],
 array([[10000,     0],
        [    0, 10000]]),
 '*')
```

4.1 a

Using Theorem 3.2, this is a ring:

1. S_1 is closed under addition: $x, y \in S_1 \Rightarrow x + y = \begin{smallmatrix} 0 & a \\ 0 & -a \end{smallmatrix} + \begin{smallmatrix} 0 & b \\ 0 & -b \end{smallmatrix} \Rightarrow x + y = \begin{smallmatrix} 0 & a+b \\ 0 & (-a)+(-b) \end{smallmatrix} = \begin{smallmatrix} 0 & a+b \\ 0 & -(a+b) \end{smallmatrix}$ which satisfies the rule

2. S_1 is closed under multiplication: $x, y \in S_1 \Rightarrow x \cdot y = \begin{smallmatrix} 0 & a \\ 0 & -a \end{smallmatrix} \cdot \begin{smallmatrix} 0 & b \\ 0 & -b \end{smallmatrix} \Rightarrow x \cdot y = \begin{smallmatrix} 0+(a)(0) & 0(b)+a(-b) \\ 0+(-a)(0) & 0b+(-a)(-b) \end{smallmatrix}$ $= \begin{smallmatrix} 0 & -ab \\ 0 & ab \end{smallmatrix}$ which satisfies the rule

3. $0_R = 0_{S_1} = \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$

4. $a \in S_1 \Rightarrow a + x = 0_R \wedge x \in S_1$ since $\begin{smallmatrix} 0 & a \\ 0 & -a \end{smallmatrix} + x = 0_{S_1} = \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \Rightarrow x = \begin{smallmatrix} 0 & -a \\ 0 & a \end{smallmatrix} \in S_1$

4.2 b

$\begin{smallmatrix} 0 & -100 \\ -100 & 0 \end{smallmatrix} \cdot \begin{smallmatrix} 0 & -100 \\ -100 & 0 \end{smallmatrix} = \begin{smallmatrix} 10000 & 0 \\ 0 & 10000 \end{smallmatrix}$ is not in the ring.

5 Question Five

No, it is not a homomorphism from the definition in 3.3.
Consider $n = \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}$ and $m = \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}$.
Then $nm = \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$ and thus $Trace(n) \cdot Trace(m) = 1 \wedge Trace(nm) = 0 \Rightarrow f(n)f(m) \neq f(nm)$.

6 Question Six

No, I don't believe f to be an isomorphism.

6.1 Lemma

We need to show that $x \in Q \wedge x\sqrt{2} = y \in Q \Rightarrow x = 0$.
If $x \neq 0$ then y can be represented as $\frac{a}{b}$ with $a, b \neq 0 \wedge a, b \in \mathbb{Z}$ and $(a, b) = 1$.
So, $x\sqrt{2} = \frac{a}{b} \Rightarrow x\sqrt{2}b = a \Rightarrow x^2(2b^2) = a^2$ and thus $2|aa$, and by applying Corollary 1.6, $2|a$.
By substituting $2c = a$ (a is divisible by 2), $x^2(2b^2) = 4c^2 \Rightarrow x^2b^2 = 2c^2$ which implies $2|(x^2b^2) \Rightarrow 2|x$ or $2|b$ (again by Corollary 1.6).
But 2 cannot divide b as well as a, since $(a, b) = 1$ and $b \neq 0$.
Thus $2|x$. So $x = 2d \Rightarrow 4d^2(2b^2) = a^2 \Rightarrow 4|a^2$, so $4e = a$, and by similar logic for b above, $4|x$.
Then, subsituting $x = 4f \Rightarrow 16f^2(2b^2) = a^2 \Rightarrow 16|a^2$. Again, $16|x$.
In fact, all range elements R of the recursive function on the natural numbers $f \ni f(1) = 2, f(n) = f(n-1)^2\{n > 1\}$ must divide x.
Thus, x must be $\sup(R)$, or 0 (everything divides zero). Obviously, though, R is unbounded on the right, so it follows $x = 0$.

6.2 Proof

$a, b \in Q(\sqrt{2}) \Rightarrow f(a) = n + m\sqrt{2} \wedge b = x + y\sqrt{2} \Rightarrow f(a) = n - m\sqrt{2} \vee f(b) = x - y\sqrt{2}$.
f is not injective; if we assume $f(a) = f(b) \Rightarrow a = b$ then $f(a) - f(b) = 0 \Rightarrow n - m\sqrt{2} - x - y\sqrt{2} = 0 \Rightarrow 0 = (n - x) - (m + y)\sqrt{2} \Rightarrow n - x = (m + y)\sqrt{2}$ and, as $n - x \in Q$ since Q is a ring and by definition, $(m + y)\sqrt{2} \in Q$ implies $(m + y) = 0$ by the lemma, thus $m = -y$. But $a = b \Rightarrow m = y$, a contradiction.