

Midterm One

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1 Question One

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In [1]: sols = lambda n: set(filter(lambda x: (x**2 + x) % n == 0, range(n)))
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In [2]: sols(5)
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Out[2]: {0, 4}
```

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In [3]: sols(6)
```

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Out[3]: {0, 2, 3, 5}
```

2 Question Two - TODO

3 Question Three

From the multiplication table we can see that e is the identity element 1_R , the zero element 0_R is z . From the addition table we find that every element is its own additive inverse.

3.1 a

The units in this ring are e, g , and the zero divisors are a, b, c, f .

3.2 b

$$3bd = 3(bd) = 3z = b$$

$$2ac = 2(ac) = 2b = b + b = z$$

$$g^{-1}f^3 = (g)(fff)(gf)(ff) = (d)(f) = d$$

$$\text{Thus, } 3bd - 2ac + g^{-1}f^3 = b - z + d = b + d = f.$$

4 Question Four

To get a hunch,

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In [1]: import numpy as np
```

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In [2]: def find_counter(matrix_generator, in_ring, search_space=range(-100, 100)):
...:     for a1 in search_space:
...:         n = matrix_generator(a1)
...:         for a2 in search_space:
...:             m = matrix_generator(a2)
...:             dot = np.dot(n, m)
...:             add = np.add(n, m)
...:
...:             if not in_ring(dot):
...:                 return (n, m, dot, '*')
...:             if not in_ring(add):
...:                 return (n, m, add, '+')
...:
...:     return None
...:
```

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In [3]: find_counter(lambda a: [[0, a], [0, -a]], lambda m: m[0][1] == -m[1][1] and m[0][0] == 0 and m[1][0] == 0)
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In [4]: In [31]: find_counter(lambda a: [[0, a], [a, 0]], lambda m: m[0][1] == m[1][0] and m[1][1] == 0 and m[0][0] == 0)
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Out[4]:
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(([0, -100], [-100, 0]),
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[[0, -100], [-100, 0]],
array([[10000,      0],
       [      0, 10000]]),
'*)')
```

4.1 a

Using Theorem 3.2, this is a ring:

1. S_1 is closed under addition: $x, y \in S_1 \Rightarrow x + y = \begin{pmatrix} 0 & a \\ 0 & -a \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & -b \end{pmatrix} \Rightarrow x + y = \begin{pmatrix} 0 & a+b \\ 0 & -(a+b) \end{pmatrix} = \begin{pmatrix} 0 & a+b \\ 0 & -(a+b) \end{pmatrix}$ which satisfies the rule
2. S_1 is closed under multiplication: $x, y \in S_1 \Rightarrow x \cdot y = \begin{pmatrix} 0 & a \\ 0 & -a \end{pmatrix} \cdot \begin{pmatrix} 0 & b \\ 0 & -b \end{pmatrix} \Rightarrow x \cdot y = \begin{pmatrix} 0+(a)(0) & 0(b)+a(-b) \\ 0+(-a)(0) & 0b+(-a)(-b) \end{pmatrix} = \begin{pmatrix} 0 & -ab \\ 0 & ab \end{pmatrix}$ which satisfies the rule
3. $0_R = 0_{S_1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
4. $a \in S_1 \Rightarrow a + x = 0_R \wedge x \in S_1$ since $\begin{pmatrix} 0 & a \\ 0 & -a \end{pmatrix} + x = 0_{S_1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 0 & -a \\ 0 & a \end{pmatrix} \in S_1$

4.2 b

$\begin{pmatrix} 0 & -100 \\ -100 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -100 \\ -100 & 0 \end{pmatrix} = \begin{pmatrix} 10000 & 0 \\ 0 & 10000 \end{pmatrix}$ is not in the ring.

5 Question Five

No, it is not a homomorphism from the definition in 3.3.

Consider $n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $m = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Then $nm = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and thus $Trace(n) \cdot Trace(m) = 1 \wedge Trace(nm) = 0 \Rightarrow f(n)f(m) \neq f(nm)$.

6 Question Six

No, I don't believe f to be an isomorphism.

6.1 Lemma

We need to show that $x \in Q \wedge x\sqrt{2} = y \in Q \Rightarrow x = 0$.

If $x \neq 0$ then y can be represented as $\frac{a}{b}$ with $a, b \neq 0 \wedge a, b \in \mathbb{Z}$ and $(a, b) = 1$.

So, $x\sqrt{2} = \frac{a}{b} \Rightarrow x\sqrt{2}b = a \Rightarrow x^2(2b^2) = a^2$ and thus $2|aa$, and by applying Corollary 1.6, $2|a$.

By substituting $2c = a$ (a is divisible by 2), $x^2(2b^2) = 4c^2 \Rightarrow x^2b^2 = 2c^2$ which implies $2|(x^2b^2) \Rightarrow 2|x$ or $2|b$ (again by Corollary 1.6).

But 2 cannot divide b as well as a , since $(a, b) = 1$ and $b \neq 0$.

Thus $2|x$. So $x = 2d \Rightarrow 4d^2(2b^2) = a^2 \Rightarrow 4|a^2$, so $4e = a$, and by similar logic for b above, $4|x$. Then, substituting $x = 4f \Rightarrow 16f^2(2b^2) = a^2 \Rightarrow 16|a^2$.

Again, $16|x$.

In fact, all range elements R of the function on the natural numbers $f \ni f(1) = 2, f(n) = f(n-1)^2n > 1$ must divide x .

Thus, x must be $\sup(R)$ or 0. Obviously R is unbounded on the right, so it follows $x = 0$.

6.2 Proof

$a, b \in Q(\sqrt{2}) \Rightarrow f(a) = n + m\sqrt{2} \wedge b = x + y\sqrt{2} \Rightarrow f(a) = n - m\sqrt{2} \vee f(b) = x - y\sqrt{2}$.

f is not injective; if we assume $f(a) = f(b) \Rightarrow a = b$ then $f(a) - f(b) = 0 \Rightarrow n - m\sqrt{2} - x - y\sqrt{2} = 0 \Rightarrow 0 = (n - x) - (m + y)\sqrt{2} \Rightarrow n - x = (m + y)\sqrt{2}$ and, as $n - x \in Q$ since Q is a ring and by definition, $(m + y)\sqrt{2} \in Q$ implies $(m + y) = 0$ by the lemma, thus $m = -y$. But for $a = b$, $m = y$, a contradiction.